

Statistical mechanics of a single particle in a multiscale random potential: Parisi landscapes in finite-dimensional Euclidean spaces

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2008 J. Phys. A: Math. Theor. 41 324009

(<http://iopscience.iop.org/1751-8121/41/32/324009>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.150

The article was downloaded on 03/06/2010 at 07:05

Please note that [terms and conditions apply](#).

Statistical mechanics of a single particle in a multiscale random potential: Parisi landscapes in finite-dimensional Euclidean spaces

Yan V Fyodorov¹ and Jean-Philippe Bouchaud²

¹ School of Mathematical Sciences, University of Nottingham, Nottingham NG72RD, UK

² Science & Finance, Capital Fund Management 6-8 Bd Haussmann, 75009 Paris, France

Received 26 November 2007, in final form 5 February 2008

Published 30 July 2008

Online at stacks.iop.org/JPhysA/41/324009

Abstract

We construct an N -dimensional Gaussian landscape with multiscale, translation invariant, logarithmic correlations and investigate the statistical mechanics of a single particle in this environment. In the limit of high dimension $N \rightarrow \infty$ the free energy of the system and overlap function are calculated exactly using the replica trick and Parisi's hierarchical ansatz. In the thermodynamic limit, we recover the most general version of the Derrida's generalized random energy model (GREM). The low-temperature behaviour depends essentially on the spectrum of length scales involved in the construction of the landscape. If the latter consists of K discrete values, the system is characterized by a K -step replica symmetry breaking solution. We argue that our construction is in fact valid in any finite spatial dimensions $N \geq 1$. We discuss the implications of our results for the singularity spectrum describing multifractality of the associated Boltzmann–Gibbs measure. Finally we discuss several generalizations and open problems, such as the dynamics in such a landscape and the construction of a generalized multifractal random walk.

PACS numbers: 64.60.De, 64.60.Al

1. Introduction

Ever since the seminal paper of Goldstein in 1969 [1], the idea of energy landscapes pervades the theoretical description of glasses, disordered systems, proteins, etc (see [2–5] and references therein). The general idea is to describe the statics and dynamics of the whole system, or one of its subparts, by a single point particle moving in a random potential, which encodes the complexity of the original system. The hope then is to be able to classify the possible classes of random potential and to establish generic, universal properties, in the spirit of random matrix theory. In this respect, the Parisi solution for spin-glasses is fascinating: it reveals that in this case the energy landscape has a surprisingly complex,

hierarchical structure of valleys within valleys, etc [6]. It is often argued that this construction is very specific not only to infinite-range spin models, but also to infinite-dimensional landscape models. In particular, the ultrametric properties of Parisi landscapes seem at first sight hardly compatible with a finite-dimensional, translation invariant random function³. In this paper, we provide an explicit construction of a Gaussian random potential in Euclidean, N -dimensional spaces, with a specific form of long-ranged correlations which reproduces all the features of Parisi landscapes. More precisely, we show that the thermodynamics of a single particle in a multiscale, logarithmically-correlated potential is exactly described by Derrida's generalized random energy model (GREM, [8]), with an arbitrary (possibly infinite) number of levels of hierarchy. Although our proof concerns, strictly speaking, the limit $N \rightarrow \infty$, we are confident that our results hold in arbitrary finite dimension $N \geq 1$. This conviction is built both on physical arguments and on the beautiful results of Carpentier and Le Doussal [9] on the monoscale version of our model in finite dimensions, which, as shown recently, match precisely the exact results of the same model when $N \rightarrow \infty$ [10].

As is well known, the Gibbs–Boltzmann measures in systems with disorder often possess the interesting property of being multifractal (see the papers [9, 11, 12]). In fact, this property is not unrelated to the multifractality of the wavefunctions in disordered electronic systems (see [13] for a comprehensive discussion of the last topic and further references). The investigation of multifractal measures of diverse origin has been a very active field of research in various branches of physics for about two decades now [13–15]. From this point, we show that our results imply, in particular, the possibility of a rather rich and unusual behaviour of the singularity spectrum describing multifractality of the Gibbs–Boltzmann measure arising in our multiscale logarithmic model.

Another closely related aspect is that the monoscale model of Carpentier and Le Doussal is known to be the building block in the construction of an exact multifractal random walk (MRW), proposed by Bacry, Muzy and Delour to describe financial time series [14]; the Boltzmann weight in one language corresponds to the local volatility in the other. So our extended multiscale model can also be interpreted as the construction of a generalized multifractal random walk (GMRW), in the same sense as the GREM generalizes the random energy model. Physically, the result is that the n th moment of the distance travelled by the random walk scales with an exponent that does not only depend on n but also on the *epoch*, i.e. the logarithm of the time lag.

The outline of the paper is as follows. In section 2 we introduce landscape models in full generality, recall the set of previously established results and discuss how they can be understood qualitatively. We then focus on the case of logarithmically-correlated landscapes, summarize the recent findings of [10] and their relation with the results of Carpentier and Le Doussal [9]. In section 3, we define precisely our multiscale random landscape model, discuss its physical motivation and detail our analytical calculations in the large dimensional $N \rightarrow \infty$ limit, where we recover exactly the GREM results. We end this section by describing the most interesting features of the multifractality spectrum of the associated Boltzmann–Gibbs measure implied by our results for the multiscaled landscapes. Finally, in section 4 we put forward several conjectures about the *dynamics* of a point particle in such landscapes, open problems and generalizations, concerning for example generalized multifractal random walks. Some more technical points are relegated to the appendices.

A short account of some of the results of the present paper was presented in [16].

³ In fact, random potentials with a hierarchical Parisi structure can be constructed in finite-dimensional space in a kind of ad hoc way, by following step by step the real-space interpretation replica symmetry breaking; see [7] for a discussion of this point.

2. Thermodynamics of a particle in a random potential

2.1. General discussion

As alluded to in the introduction, the ‘toy-model’ of a classical particle in a random potential exhibits a rich variety of behaviour which mimics many of the dynamical and thermodynamical properties of glassy systems. The model is defined as follows: the position of the particle, confined inside an N -dimensional spherical box of radius L , is described by the coordinate vector $\mathbf{r} = (r_1, \dots, r_N)$, $|\mathbf{r}| \leq L$. It feels a random potential $V(\mathbf{r})$, which we conventionally choose to be Gaussian distributed with zero mean, and with covariance chosen to be isotropic, translation invariant and with a well-defined large N -limit,

$$\langle V(\mathbf{r}_1)V(\mathbf{r}_2) \rangle_V = Nf\left(\frac{1}{2N}(\mathbf{r}_1 - \mathbf{r}_2)^2\right). \quad (1)$$

In equation (1) and henceforth the notation $\langle \dots \rangle_V$ stands for an ensemble average over the random potential, and $f(u)$ is a function of order unity belonging to the so-called class \mathcal{D}_∞ described in detail, e.g., in the book by Yaglom [17]. The functions $f(u) \in \mathcal{D}_\infty$ are such that they represent covariances of an isotropic random field for *any* spatial dimension $N \geq 1$. There are two essentially different types of such functions. The first type corresponds to genuine isotropic random fields, and those $f(u)$ are characterized by a non-negative normalizable ‘spectral density function’ $\tilde{f}(k) \geq 0, k \geq 0$ in terms of which $f(u)$ is represented as (see [17, p 354]),

$$f(u) = \int_0^\infty e^{-k^2u} \tilde{f}(k) dk, \quad f(0) = \int_0^\infty \tilde{f}(k) dk < \infty. \quad (2)$$

In particular, $f(u)$ is decreasing and convex, i.e. satisfies $f'(u) < 0, f''(u) > 0 \forall u \geq 0$, and in addition $f'(u \rightarrow \infty) = 0$. Here and below the number of dashes indicates the order of derivatives taken. A few important families of such functions listed in [17] are, e.g., (i) $f(u) = Ce^{-au^\gamma}, 0 \leq \gamma \leq 1, C > 0, a > 0$ (ii) $f(u) = C/(a+u)^\gamma, \gamma > 0, C > 0, a > 0$ and (iii) $f(u) = C(au)^{\gamma/2} K_\gamma(\sqrt{au}), \gamma > 0, C > 0, a > 0$, where $K_\gamma(x)$ stands for the modified Bessel (a.k.a. Macdonald) function. In the physical literature, the random fields of that type are frequently called potentials with short-ranged (SR) correlations.

The second type of covariances occurs in the situation when the normalization integral $\int_0^\infty \tilde{f}(k) dk$ diverges. It corresponds to long-ranged (LR) random fields with isotropic increments also known as *locally* isotropic random fields (see, e.g., [17, p 438]). The spectral density function now must satisfy the condition $\int_0^\infty \frac{k^2}{k^2+1} \tilde{f}(k) dk < \infty$ which allows one to prove that in any dimension $N \geq 1$ there exists a random field whose *structure function* $\frac{1}{2}\langle (V(0) - V(\mathbf{r}))^2 \rangle_V = f(0) - f(u)$ is given by

$$f(0) - f(u) = \int_0^\infty dk(1 - e^{-k^2u}) \tilde{f}(k) dk + Au, \quad A \geq 0. \quad (3)$$

In what follows we will impose an additional requirement $f'(u \rightarrow \infty) = 0$, which ensures $A = 0$ (no external driving force acting on the particle). It is also easy to convince oneself that in the present model the difference between the covariance and the structure function, i.e. the value of $f(0)$, is immaterial for the free-energy calculations. The most widely-known example of the locally isotropic LR field is the so-called self-similar random field, see [17, p 441], characterized by the spectral density $\tilde{f}(k > 0) = k^{-2\gamma-1}, 0 < \gamma < 1$. The corresponding covariance behaves as

$$f(u) = f(0) - C_\gamma u^\gamma. \quad (4)$$

In particular, for $N = 1$ and $\gamma = 1/2$ this is the example of a simple Brownian motion for the potential, corresponding to the celebrated Sinai model.

After specifying in detail the class of random potentials involved in our construction, let us turn to thermodynamics of the model characterized by the following partition function and the corresponding free energy:

$$F = -T \langle \ln Z \rangle_V, \quad Z = \int_{|\mathbf{r}| \leq L} \exp -\beta V(\mathbf{r}) \, d\mathbf{r}, \quad (5)$$

where $\beta = \frac{1}{T}$ stands for the inverse temperature. A variant of the model consists in replacing the spherical box $|\mathbf{r}| \leq L$ by a confining harmonic potential $-\mu \mathbf{r}^2/2$. Such a model has been studied extensively since the mid-1980s. It was originally proposed in $N = 1$ dimension as a toy model for a randomly pinned domain wall [18] or of a directed polymer in a random potential [19] and studied using a variety of methods [9, 20–22], some of them being exact. Another case where analytical calculations can be performed is the high-dimensional limit $N \rightarrow \infty$ [23, 24], where a Gaussian variational ansatz with replica symmetry breaking becomes exact. One finds that the nature of the low-temperature phase is essentially dependent on the behaviour of the covariance at large distances [23, 24]. Namely, for typical short-ranged correlated potential the description of the low-temperature phase was found to require the so-called one-step replica symmetry breaking (1RSB) scheme of Parisi. In contrast, for the potentials growing as u^γ , see equation (4), the full infinite-hierarchy replica symmetry breaking (FRSB) scheme has to be used. In fact, this problem was reconsidered recently in [10] in much detail using an alternative method that directly focuses on the degrees of freedom relevant in the limit $N \rightarrow \infty$, and employs the Laplace (aka saddle-point) method for evaluating the integrals. In the limit $N \rightarrow \infty$, one actually finds a true phase transition as a function of temperature provided the radius of the confining sphere L is scaled as $R\sqrt{N}$. The effective size $R < \infty$ (which is accidentally just half of the length of an edge of the cube inscribed in this sphere) is then used as the main control parameter of the model. The chosen scaling $L \sim \sqrt{N}$ formally stems from the property of the argument of the correlation function equation (1) to become of order of unity for separations of order of \sqrt{N} . More importantly, it simultaneously ensures that the volume $V_L = \pi^{N/2} \frac{L^N}{\Gamma(N/2+1)}$ of our spherical sample retains in the limit $N \gg 1$ the natural scaling with size R and dimension N : $\ln V_L = N \ln (R/R_0) + \text{smaller terms}$, with R_0 being a constant of order of unity. Such a behaviour is essential since the phase transition is physically induced by a competition between entropic effects, which tend to delocalize the particle over the sphere, and the minima of the random potential which tend to attract, and possibly to localize the particle over a *finite* number of favourable sites. In the short-range case, the number of effectively independent ‘sites’ is of the order of $V_L \propto R^N$ which ensures, thanks to the N factor in front of $f(u)$ in equation (1), that the minimum of the Gaussian potential scales as $\sqrt{N} \sqrt{\ln R^N} \propto N$. This indeed can compete with the entropy of the order $N \ln R$. All these arguments demonstrate that indeed R is the most natural measure of the sample size.

The fact that a true thermodynamic transition exists for a finite sample size $R < \infty$ is obviously a somewhat pathological feature of the limit of infinite dimension $N \rightarrow \infty$ taken first. Indeed, in that limit the total number of thermodynamic degrees of freedom is infinite even for finite R . At the same time, at any finite spatial dimension $N < \infty$ phase transitions only may occur in the thermodynamic limit of infinite sample size $R \rightarrow \infty$. From this point of view it is natural to inspect the $R \rightarrow \infty$ behaviour of the transition temperature $T_c(R)$ at which the system experiences a continuous breakdown of replica symmetric solution. For the present model it reads, according to [10],

$$T_c(R) \approx R^2 \sqrt{f''(R^2)} \quad (6)$$

in agreement with a similar result for the confining quadratic potential case [23, 24]. Therefore, one finds that $T_c(R)$ tends for $R \rightarrow \infty$ either to zero in the SR case⁴, or to infinity in the LR case. A more detailed analysis shows correspondingly that the large R behaviour of the free energy is $F(T)|_{R \rightarrow \infty} \sim -T \ln R^N$ in the SR case where entropy dominates, and $F(T)|_{R \rightarrow \infty} \sim -NR^\gamma$ in the LR case where the deepest minimum dominates.

2.2. Logarithmically-correlated potentials

From equation (6) above, $T_c(R)|_{R \rightarrow \infty}$ appears to have a well-defined limit when $f''(u) \sim u^{-2}$ for large u , corresponding to a logarithmically growing correlation function of random potential, a case overlooked in previous studies [23, 24]. The peculiarities of that case can be traced in a few different ways. To this end, it is appropriate to mention a precise mathematical criterion proposed recently in [10] to classify statistical mechanics behaviour induced by SR versus LR correlated potentials. The criterion uses the notion of the so-called Schwarzian derivative $\{f'(u), u\} = -S(u)/[f''(u)]^2$, where $S(u)$ is expressed in terms of $f(u)$ as

$$S(u) = \frac{3}{2}[f'''(u)]^2 - f''(u)f''''(u). \quad (7)$$

In terms of $S(u)$ it was demonstrated that:

- Any potential whose covariance function satisfies the condition $S(u) > 0 \forall u \geq 0$ must have a 1RSB low-temperature phase. It is easy to check that such a situation includes, in particular, the standard families of the SR potentials (i) and (ii) listed after equation (2). For a general case of SR fields with covariances defined via equation (2) one always has $S(u) > 0$ for large enough values of u , which is the most essential range in the thermodynamic limit.
- Any potential whose correlation function satisfies the condition $S(u) < 0 \forall u \geq 0$ must necessarily have the FRSB low-temperature phase. This condition holds for the standard LR correlation functions of the type equation (4), i.e. for $f(u) = f_0 - g^2(u + a^2)^\gamma$, with $0 < \gamma < 1$ and $f_0 - g^2 a^{2\gamma} > 0$. It is natural to conjecture that typical LR random fields with independent increments should be of this type for large enough u .

Clearly, the above criterion naturally singles out as a special marginal case random potentials satisfying $S(u) = 0$. The only function satisfying this condition globally, i.e. for all $u \geq 0$, and satisfying also the requirement $f'(u \rightarrow \infty) = 0$ is indeed given by a *logarithmic* correlation function, of the type considered by Carpentier and Le Doussal in finite dimensions [9],

$$f(u) = f_0 - g^2 \ln(u + a^2), \quad (8)$$

where g and a are given constants, and f_0 is such that $f(0) = f_0 - g^2 \ln(a^2) > 0$.⁵ Let us stress that the expression equation (8) is a legitimate covariance function belonging to the \mathcal{D}_∞ class of LR locally isotropic fields, equation (3). Indeed, it corresponds to the spectral density of the form $\tilde{f}(k) = \frac{2g^2}{k} e^{-a^2 k^2}$, which satisfies the required condition $\int_0^\infty \frac{k^2}{k^2+1} \tilde{f}(k) dk < \infty$.

⁴ The situation is slightly more complicated, as for SR case there exists another temperature $T_1(R) > T_c(R)$ where the system experiences a discontinuous breakdown of replica symmetry. However one can show that $T_1(R \rightarrow \infty) \rightarrow 0$ as well [10].

⁵ As noted above, the value of $f(0)$ is irrelevant for the thermodynamics of the system, so that f_0 can be dropped from the calculations. We systematically disregard such constants in the rest of the paper.

In such a case of logarithmically-correlated potentials the solution found in [10] has features of both the SR-1RSB and LR-FRSB regimes. The critical temperature T_c remains finite for large systems, and is given by

$$T_c(R \rightarrow \infty) = g. \quad (9)$$

Physically, the minima of the potential now typically behave as

$$V_{\min}(R) \sim -g\sqrt{N}\sqrt{2\ln R^N}\sqrt{\ln(a^2 + R^2)} \sim -2gN \ln R, \quad (10)$$

while the entropy contribution is $-TN \ln R$, suggesting that indeed some change of physics should take place when $T \sim g$ (see [9] for further elaboration of this argument).

Furthermore, it is easy to check that the free-energy expression found in [10] for arbitrary R and a reduces in the limit $R \gg a$ to that of the famous random energy model [27],

$$-\frac{1}{N}F(T)|_{R \gg a} = \begin{cases} T(1 + g^2/T^2) \ln R, & T > T_c \\ 2g \ln R, & T < T_c. \end{cases} \quad (11)$$

Interestingly, these results coincide *precisely* with the renormalization group results of Carpentier and Le Doussal in finite dimensions (up to a rescaling of their coupling constant $g = \sqrt{\sigma}$ by a factor \sqrt{N} , as indicated by equation (1)). The interpretation is the same as for the REM: below T_c , the partition function becomes dominated by a finite number of sites where the random potential is particularly low, and where the particle ends up spending most of its time [25, 26]. For a more quantitative description of the particle localization, useful in the following, it is natural to employ the overlap function defined as the mean probability for two *independent* particles placed in the same random potential to end up at a given distance to each other. Denoting the scaled Euclidean distance (squared) between the two points in the sample as \mathcal{D} , and employing the Boltzmann–Gibbs equilibrium measure $p_\beta(\mathbf{r}) = \frac{1}{Z(\beta)} \exp -\beta V(\mathbf{r})$ the above probability in thermodynamic equilibrium should be given by

$$\pi(\mathcal{D}) = \left\langle \int_{|\mathbf{r}_1| < L} d\mathbf{r}_1 p_\beta(\mathbf{r}_1) \int_{|\mathbf{r}_2| < L} d\mathbf{r}_2 p_\beta(\mathbf{r}_2) \delta\left(\mathcal{D} - \frac{1}{2N}|\mathbf{r}_1 - \mathbf{r}_2|^2\right) \right\rangle_V, \quad (12)$$

where here and henceforth δ denotes the Dirac's δ -function. The disorder averaging in (12) can be calculated following the same standard steps of the replica approach as the free energy itself. For convenience of the reader we sketch the procedure for the present model in appendix A.

With the function $\pi(\mathcal{D})$ in hand we can ask, in particular, what is the probability for the particle in logarithmically-correlated potential to end up at $\mathcal{D} = O(a^2)$, i.e. at a distance of order of the small cutoff scale. The answer turns out to be zero in the high-temperature phase $T > T_c$, confirming the particle delocalization over the sample. In contrast, in the lower temperature phase $T < T_c$ the probability is finite: $\pi(O(a^2)) = 1 - T/T_c$, again in full agreement with REM calculation [9, 27–29].

A logarithmic growth of the variance of the potential might look an academic oddity, but in fact is not, and appears naturally in various systems of actual physical interest. We warmly recommend the paper of Carpentier and Le Doussal [9], which discusses in detail the connection to many other interesting and important physical problems, like directed polymers in random environment [30], or a quantum particle in a random magnetic field [11]. In the following section, we introduce and study a very natural, multiscale generalization of this model.

3. A multiscale logarithmic potential

3.1. Motivation and definition of the model

The main observation of the present paper is that the above picture, despite looking rather complete, still misses a rich class of possible behaviour that survives in the thermodynamic limit $R \rightarrow \infty$. Namely, given any increasing positive function $\Phi(y)$ for $0 < y < 1$, we demonstrate below that if one considers potential correlation functions $f(u)$ which take the following scaling form

$$f(u) = -2 \ln R \Phi \left(\frac{\ln(u + a^2)}{2 \ln R} \right), \quad 0 \leq x < R^2, \quad (13)$$

the thermodynamics of our system in the limit $R \rightarrow \infty$ is precisely equivalent to that of celebrated Derrida's generalized random energy model (GREM) [8]. The REM-like case discussed above turns out to be only a (rather marginal) representative of this class corresponding to specific choice of the scaling function $\Phi(y) = g^2 y$.

Let us explain the motivation of the above form, which will make the physical interpretation of the results (as well as some technical calculations) quite transparent. The idea is to write $V(\mathbf{r})$ as a (possibly infinite) sum of K independent Gaussian potentials,

$$V(\mathbf{r}) = \sum_{i=1}^K V_i(\mathbf{r}), \quad (14)$$

each with a simple logarithmic covariance as in (8),

$$\langle V_i(\mathbf{r}_1) V_j(\mathbf{r}_2) \rangle_V = \delta_{i,j} N f_i \left(\frac{1}{2N} (\mathbf{r}_1 - \mathbf{r}_2)^2 \right), \quad f_i(u) = -g_i^2 \ln(u + a^2 + a_i^2), \quad (15)$$

each with its own strength constant g_i and small-scale cutoff a_i , which we choose to grow as a power law of the system size⁶: $a_i = R^{v_i}$ with $0 \leq v_i \leq 1$. Taking the continuum limit $K \rightarrow \infty$ with a certain density $\rho(v)$ of exponents v_i , we end up with

$$f(u) = - \int_0^1 \rho(v) g^2(v) \ln(u + a^2 + R^{2v}) dv, \quad 0 \leq x \leq R^2. \quad (16)$$

Now, introducing $u + a^2 \equiv R^{2y}$ and identifying with equation (13) in the $R \rightarrow \infty$ limit, we find that the function Φ has the following representation:

$$\Phi(y) = y \int_0^y \rho(v) g^2(v) dv + \int_y^1 v \rho(v) g^2(v) dv, \quad (17)$$

the previous REM case corresponding to $\rho(v) = \delta(v)$. Note also that in this representation, $\Phi'(y) = \int_0^y \rho(v) g^2(v) dv \geq 0$, and also $\Phi''(y) \geq 0$. The main result of this work is the following: depending on the nature of the spectrum of the exponents v , discrete or continuous, we will recover, in the thermodynamic limit, either the free energy of the original GREM with discrete hierarchical structure, or of its continuous hierarchy analogue (see (32)) analysed recently in much detail by Bovier and Kurkova [31], and appearing also in earlier studies of random heteropolymers [32].

The physical interpretation of our results also generalize the discussion of the previous section 2.2, in a natural and, we believe, rather beautiful way. Instead of one localization transition temperature T_c where the particle chooses a finite number of 'blobs' of size $O(a)$ where the potential is particularly deep, there appears K different transition temperatures,

⁶ One could in fact multiply this power-law behaviour by a slow function of R with no impact on the following results in the limit $\ln R \rightarrow \infty$.

where the particle localizes on finer and finer length scales. The largest transition temperature T_1 corresponds to a condensation of the Boltzmann–Gibbs weight inside a few blobs of large size $O(R)$, but the particle is still completely delocalized *inside* each blob. As the temperature is reduced, the REM condensation takes place over smaller blobs of size $O(R^\nu)$ inside each already occupied large blobs, and this scenario repeats itself as the temperature is reduced, each time ‘zooming’ in on a smaller scale⁷. To see this most clearly we quote the simplest example beyond REM, the two-scale logarithmic model characterized by the density of exponents $g^2(\nu)\rho(\nu) = g_2^2\delta(\nu) + g_1^2\delta(\nu - \nu_1)$, with $0 < \nu_1 < 1$. The system turns out to be described by two different critical temperatures $T_1 = \sqrt{g_1^2 + g_2^2} > T_2 = g_2$. As we will be able to show, see equation (45), in our model the probability $\Pi(\nu)$ for two particles to be found at a distance $D = O(R^{2\nu})$ apart is given by

$$\Pi(\nu) = \begin{cases} \delta(\nu - 1), & T > T_1 \\ \left(1 - \frac{T}{T_1}\right)\delta(\nu - \nu_1) + \frac{T}{T_1}\delta(\nu - 1), & T_2 < T \leq T_1, \\ \left(1 - \frac{T}{T_2}\right)\delta(\nu) + \left(\frac{T}{T_2} - \frac{T}{T_1}\right)\delta(\nu - \nu_1) + \frac{T}{T_1}\delta(\nu - 1), & 0 \leq T \leq T_2. \end{cases} \quad (18)$$

The first two lines reproduce the former REM scenario, with T_1 standing for T_c and the scale $a_1 = O(R^{\nu_1})$ playing for $T > T_2$ the role of the lowest discernible cutoff scale. The last line describes quantitatively the ‘zooming in’ from the scale $a_1 = O(R^{\nu_1})$ to the even smaller cutoff scale $a = O(R^0)$, which becomes discernible below $T = T_2$ and dominates more and more when $T \rightarrow 0$.

3.2. Analytical results for $N \rightarrow \infty$

We aim to compute the equilibrium free energy per degree of freedom of our model, $F_\infty = \lim_{N \rightarrow \infty} F_N/N$, where F_N is defined in equation (5). The disorder average is performed in a standard way using the replica trick. The replicated partition function $\langle Z^n \rangle_V$ is evaluated exactly for $1 \leq n \leq N - 1$ in the large- N limit by the Laplace method, after exploiting a high symmetry of the integrand stemming from the symmetry of the correlation function equation (1). The replica limit $n \rightarrow 0$ is then performed in the standard framework of the Parisi hierarchical ansatz. The details of the corresponding analysis can be found in [10], and we give below a summary of the most essential formulae.

From the point of view of the Schwarzian derivative criterion recalled above, our models (13) and (16) are such that for any finite $R < \infty$, the low-temperature phase is characterized by *continuous* FRSB Parisi pattern with infinite level of hierarchy. This holds invariably, even when we make the choice of a discrete set of $K \geq 2$ distinct exponents $0 < \nu_K < \dots < \nu_1 < 1$. Indeed, the function $S(u)$ defined in equation (7) when calculated from equation (16) reads

$$S(u) = -3 \int \int_0^1 \frac{g^2(\nu)\rho(\nu)g^2(\nu')\rho(\nu')}{(u + s_\nu^2)^2(u + s_{\nu'}^2)^2} \left[\frac{1}{(u + s_\nu^2)} - \frac{1}{(u + s_{\nu'}^2)} \right]^2 d\nu d\nu', \quad (19)$$

where we have used the short-hand notation $s_\nu^2 = a^2 + R^{2\nu}$. The expression in the right-hand side of (19) is manifestly strictly negative, apart from the discrete $K = 1$ case when it is zero. Only in the thermodynamic limit $R \rightarrow \infty$ shall we find that $S(u) \rightarrow 0$ for most values of u . In this limit the system effectively recovers a GREM structure corresponding to the replica symmetry breaking pattern with K levels of hierarchy⁸.

⁷ See [45] for a related discussion of the idea that temperature plays the role of a microscope in the context of spin-glasses.

⁸ An alternative construction of multiscale logarithmic landscapes which show replica symmetry breaking with exactly K levels of Parisi hierarchy for *finite* R is proposed in appendix B.

For finite R , the low-temperature phase is therefore characterized by the existence of a non-trivial, non-decreasing function $x(q)$, $q \in [q_0, q_k]$, with the two parameters q_0 and q_k satisfying the inequality $0 \leq q_0 \leq q_k \leq q_d \equiv R^2$. The corresponding F_∞ can be written in terms of only those two parameters, see equation (58) of [10], without explicit reference to $x(q)$. Here we find it more convenient to introduce, along the line of the physical discussion given above, two characteristic ‘blob’ sizes (actually size squared) $d_{\min} = R^2 - q_k$, $d_{\max} = R^2 - q_0$ in terms of which

$$F_\infty = -\frac{T}{2} \ln [2\pi e d_{\min}] + \frac{1}{2T} [f(d_{\min}) - f(0) - d_{\min} f'(d_{\min})] + \frac{f'(d_{\max})}{\sqrt{f''(d_{\max})}} - \int_{d_{\min}}^{d_{\max}} \sqrt{f''(u)} \, du, \quad (20)$$

where $d_{\min} \leq d_{\max}$ can be found for a given temperature T from the equations

$$0 \leq d_{\min} = \frac{T}{\sqrt{f''(d_{\min})}}, \quad d_{\max} = R^2 + \frac{f'(d_{\max})}{f''(d_{\max})} \leq R^2. \quad (21)$$

Finally, the Parisi order-parameter function, which takes the values between 0 and 1 and is the main measure of the ultrametricity in the phase space, has the following shape:

$$x(d) = -\frac{T}{2} \frac{f'''(d)}{[f''(d)]^{3/2}}, \quad \forall d \in [d_{\min}, d_{\max}]. \quad (22)$$

where again we found convenient to perform the overall change $q \rightarrow d = R^2 - q$ in comparison with notations used in [10]. This function must be now *non-increasing*, as follows from relating its derivative to the probability $\pi(d)$ introduced earlier in equation (12), see the relation equation (A.5). Technically, this property is precisely ensured by negativity of the Schwarzian derivative of $f(d)$, see discussions around equation (7).

The above solution is valid for the temperature range $0 \leq T \leq T_c$, where the critical (or de Almeida–Thouless) temperature T_c is given in terms of the largest blob size d_{\max} as

$$T_c = d_{\max} \sqrt{f''(d_{\max})}. \quad (23)$$

Above this temperature the solution is replica symmetric, corresponding to a delocalized phase for the particle: no particular region dominates the partition function. The corresponding free energy is given by

$$F_\infty = -\frac{T}{2} \ln [2\pi d_s] + \frac{1}{2T} [f(d_s) - f(0)] - \frac{T}{2} \frac{R^2}{d_s},$$

where d_s satisfies

$$d_s = R^2 + \frac{d_s^2}{T^2} f'(d_s). \quad (24)$$

We now consider specifically a correlation functions $f(u)$ of the form (13). In what follows we will use the convenient notations $z = (2 \ln R)^{-1}$ and $y = z \ln(u + a^2)$. As noted above, our multiscale logarithmic model ensures that $\Phi'(y) \geq 0$ and $\Phi''(y) \geq 0$ for any $0 < y < 1$. We also will assume in our analysis below that the function $\Phi''(y)$ is finite ($0 < \Phi''(y) < \infty$) and differentiable, but later on will relax those conditions. Simple differentiation gives

$$f'(u) = -\frac{1}{u + a^2} \Phi'(y), \quad f''(u) = \frac{1}{(u + a^2)^2} [\Phi'(y) - z \Phi''(y)]. \quad (25)$$

Our first goal is to find the largest blob size d_{\max} from the second equation in (21), and then to determine the critical temperature T_c . Introduce the scaling variable $y_{\max} = z \ln(d_{\max} + a^2)$ and using (25), we obtain the following equation determining y_{\max} :

$$e^{(1-y_{\max})/z} = (1 - a^2 e^{-y_{\max}/z}) \left[1 + \frac{\Phi'(y_{\max})}{\Phi'(y_{\max}) - z\Phi''(y_{\max})} \right]. \quad (26)$$

Since we are interested in the thermodynamic limit $z \rightarrow 0$, we can look for a solution $y_{\max}(z)$ as a power series of z . One immediately checks that $y_{\max}(z) = 1 - z \ln 2 + O(z^2)$. This implies that the largest blob size is of the order of the system radius: $d_{\max} \approx R^2/2 \gg a^2$ for $R \rightarrow \infty$. Equations (25) and (23) then yield the critical temperature given in the thermodynamic limit, which simply reads

$$T_c = \sqrt{\Phi'(1)}. \quad (27)$$

Physically, at T_c , the space breaks up blobs of size $o(R)$ and only a finite number of these blobs are visited by the particle. However, within each blob, all sites are more or less equivalent. Now we can treat along the same lines the first equation in (21) to determine the smallest blob size d_{\min} for $T < T_c$. It can again be conveniently written in terms of the scaling variable $y_{\min} = z \ln(d_{\min} + a^2)$, such that,

$$T = (1 - a^2 e^{y_{\min}/z}) \sqrt{\Phi'(y_{\min}) - z\Phi''(y_{\min})}. \quad (28)$$

This equation determines $y_{\min} \geq 0$ for any temperature $T < T_c$ and sample size $R = \exp[1/2z]$. In the thermodynamic limit $z \rightarrow 0$, it is again natural to look for a solution y_{\min} as a power series of z , in which we only retain the first two terms: $y_{\min} = \nu_* + cz + O(z^2)$. Assuming self-consistently that the solution corresponds to $\nu_* > 0$, we see that the first factor in (28) can be replaced with unity with exponential accuracy. Due to our assumption on differentiability of the function $\Phi'(y)$ we expand around $y = \nu_*$, and after a simple calculation find $c = 1$. This means that d_{\min} behaves like $d_{\min} = eR^{2\nu_*}$ for $R \rightarrow \infty$, where ν_* satisfies the equation

$$T^2 = \Phi'(\nu_*). \quad (29)$$

Since the function $\Phi'(y)$ is monotonously increasing for $y > 0$, we find that in the limit $R \rightarrow \infty$ (i.e. $z \rightarrow 0$), equation (29) must have a unique solution $1 > \nu_*(T) > 0$ in the range of temperatures $\sqrt{\Phi'(0)} = T_{\min} < T < T_c = \sqrt{\Phi'(1)}$. In this regime, $d_{\min} \ll d_{\max}$. Physically, sites within blobs of size d_{\min} or smaller are not resolved by the particle, which visits all of them more or less equally.

Now we can easily find the free energy F_∞ in the thermodynamic limit $z \rightarrow 0$. In particular, note that equation (25) implies that the last term in equation (20) can be conveniently written as

$$I = - \int_{d_{\min}}^{d_{\max}} [(u + a^2)^2 f''(u)]^{1/2} \frac{du}{u + a^2} = - \frac{1}{z} \int_{y_{\min}}^{y_{\max}} [\Phi'(y) - z\Phi''(y)]^{1/2} dy. \quad (30)$$

In the temperature range $T_{\min} < T \leq T_c$ we can substitute here $y_{\min} = \nu_*(T) + z$ and $y_{\max} = 1 - z \ln 2$, and expand in z up to linear terms. This gives

$$I = -2 \ln R \int_{\nu_*}^1 \sqrt{\Phi'(y)} dy + T_c(1 + \ln 2). \quad (31)$$

In the same way we evaluate the remaining terms in equation (20), and finally find the leading and the subleading terms for the equilibrium free energy: $F_\infty = \ln R\mathcal{F} + \delta F$, where the leading term coefficient \mathcal{F} is given by

$$- \mathcal{F}/T = \nu_*(T) + \frac{[\Phi(\nu_*) - \Phi(0)]}{T^2} + \frac{2}{T} \int_{\nu_*}^1 \sqrt{\Phi'(y)} dy, \quad T_{\min} < T \leq T_c \quad (32)$$

and the correction term is

$$-\delta F/T = \ln \sqrt{2\pi} + 1 - \frac{T_c}{T} \ln 2 - \frac{T_{\min}^2}{T^2} \ln a, \quad T_{\min} < T \leq T_c. \quad (33)$$

For $T > T_c$ the solution of (24) in the limit $a \ll R \rightarrow \infty$ is given by

$$d_s = R^2 \frac{T^2}{T^2 + T_c^2}, \quad (34)$$

and substituting this into (24) we find that the free-energy components are given by

$$-\mathcal{F}/T = 1 + \frac{[\Phi(1) - \Phi(0)]}{T^2}, \quad T > T_c, \quad (35)$$

$$-\delta F/T = \ln \sqrt{2\pi} + \frac{1}{2} \left(1 + \frac{T_c^2}{T^2}\right) \left[1 - \ln \left(1 + \frac{T_c^2}{T^2}\right)\right] - \frac{T_{\min}^2}{T^2} \ln a. \quad (36)$$

Finally, the analysis should be reconsidered for $T < T_{\min} = \sqrt{\Phi'(0)}$, where $d_{\min} \sim a^2$ and the particle localizes even on the smallest scales $O(a)$. Indeed, assuming that generically $\Phi''(0) < \infty$ it is easy to see from equation (29) that $\nu_* \approx (2T_{\min}/\Phi''(0))(T - T_{\min}) \rightarrow 0$ as $T \rightarrow T_{\min}$. The solution $d_{\min} = eR^{2\nu_*}$ is therefore invalid for $T < T_{\min}$ where ν_* stays identically zero. Using equations (21) and (25) we find after a straightforward calculation the following correct solution to equation (28) as $z \rightarrow 0$ in this range of temperatures:

$$d_{\min} = a^2 \frac{T}{T_{\min} - T}, \quad 0 \leq T < T_{\min}, \quad (37)$$

showing that T_{\min} is indeed a delocalization transition of the REM type, above which the minimum blob size becomes much larger than the small-scale cutoff a . The leading term in the free energy is given by $\nu_* \rightarrow 0$ limit of (32),

$$\mathcal{F} = -2 \int_0^1 \sqrt{\Phi'(y)} dy, \quad 0 \leq T \leq T_{\min}, \quad (38)$$

Free energy corrections can also be computed, and for all $T < T_{\min}$ are given by

$$-\delta F/T = \ln \sqrt{2\pi} + \frac{T_{\min}}{2T} - \frac{T_c}{T} \ln 2 + \left(1 - 2\frac{T_{\min}}{T}\right) \ln a + \frac{1}{2} \left(1 + \ln \frac{T}{T_{\min}}\right) - \frac{1}{2} \left(1 - \frac{T_{\min}}{T}\right)^2 \ln \left(1 - \frac{T}{T_{\min}}\right). \quad (39)$$

Comparing equations (39) and (33) we see that the correction term is continuous at the delocalization transition point $T = T_{\min}$.

Last but not least, we can determine the thermodynamic limit of the order-parameter function $x(d)$ given by equation (22), which determines in a precise way how the particle localizes on different scales. To leading order in z we find $f'''(u) = -2\Phi'(y)/u^3$ with $y = z \ln(u + a^2)$. Introduce again the scaling variable $\nu = \frac{\ln d}{2 \ln R}$, $d \in [eR^{2\nu_*}, R^2/2]$. Denoting the Parisi order-parameter function $x(d)$ expressed in terms of the new variable ν as $X(\nu)$ we see that such function assumes the limiting form

$$X(\nu) = \frac{T}{[\Phi'(\nu)]^{1/2}}, \quad \forall \nu \in [\nu_*, 1]. \quad (40)$$

This completes our solution of the problem for the case of continuous function $\Phi(y)$. At this point it is rather informative to consider the case of a discrete spectrum of exponents ν , corresponding to K superimposed logarithmic potentials with

$$g^2(\nu)\rho(\nu) = \sum_{i=1}^K g_i^2 \delta(\nu - \nu_i), \quad 0 < \nu_K < \nu_{K-1} < \dots < \nu_1 < \nu_0 = 1, \quad (41)$$

with $\delta(u)$ standing for the Dirac delta-functions. The corresponding $\Phi'(y)$ consists of steps: $\Phi'(y) = \sum_{i=1}^K g_i^2 \theta(y - \nu_i)$. A simple consideration shows that our earlier analysis for the values of d_{\max} and the critical temperature T_c still hold for such a case, so $d_{\max} = R^2/2$, and $T_c = [\Phi'(1)]^{1/2} = \sqrt{g_1^2 + g_2^2 + \dots + g_K^2}$. Equation (21) used to determine $d_{\min} = R^{2y_{\min}} - a^2$ now takes the following form:

$$T^2 = \sum_{i=1}^K g_i^2 \frac{1 - a^2 e^{-y_{\min}/z}}{1 + e^{(\nu_i - y_{\min})/z}}, \quad z = \frac{1}{2 \ln R}. \quad (42)$$

A little thought shows that the solution should always be in the form $y_{\min} = \nu_p + c_p z$ for small z , where the index p runs successively through the values $1, \dots, K$ when decreasing temperature from T_c towards $T = 0$. Introducing a decreasing sequence of characteristic temperatures $T_p = \sqrt{\sum_{i=p}^K g_i^2}$, we find the coefficients c_p and the index ν_p for a given temperature:

$$y_{\min} = \nu_p + z \ln \frac{T^2 - T_{p+1}^2}{T_p^2 - T^2}, \quad T_{p+1} < T < T_p. \quad (43)$$

Thus, the value of y_{\min} jumps (and thus the size of the smallest frozen blobs d_{\min}) when crossing each of the temperatures T_p , $p = 1, 2, \dots, K$ with the highest one being $T_1 = T_c$. Since T_p and ν_p decrease as p increases, it is clear that the order-parameter function $X(\nu)$ for a given temperature $T < T_c$ is step-wise constant with jumps at each ν_p ; the smaller ν (i.e. the smaller the size of the blobs), the larger $X(\nu)$, meaning that the condensation effect on the scale ν is weaker and the Boltzmann weight becomes delocalized for scales such that $X(\nu) \geq 1$. Explicitly, in the temperature range $T_{p+1} < T \leq T_p$ we find

$$X(\nu) = 1 + \left(\frac{T}{T_p} - 1\right) \theta(\nu - \nu_p) + \left(\frac{T}{T_{p-1}} - \frac{T}{T_p}\right) \theta(\nu - \nu_{p-1}) + \dots + \left(\frac{T}{T_1} - \frac{T}{T_2}\right) \theta(\nu - \nu_1) - \frac{T}{T_1} \theta(\nu - 1). \quad (44)$$

Invoking the relation equation (A.5) we then see that the probability $\Pi(\nu)$ for two independent particles to end up at an ultrametric separation ν is given for the present model by

$$\Pi(\nu) = \left(1 - \frac{T}{T_p}\right) \delta(\nu - \nu_p) + \left(\frac{T}{T_p} - \frac{T}{T_{p-1}}\right) \delta(\nu - \nu_{p-1}) + \dots + \left(\frac{T}{T_2} - \frac{T}{T_1}\right) \delta(\nu - \nu_1) + \frac{T}{T_1} \delta(\nu - 1), \quad (45)$$

as long as $T_{p+1} < T \leq T_p$, $p = 1, 2, \dots, K$. In particular, in the simplest case of the two-scale model $K = 2$ assuming $0 = \nu_2 < \nu_1 < 1$ for the exponents, we obtain the probability distribution quoted in equation (18).

The expressions for (y_{\min}, y_{\max}) suffice to calculate the free-energy expression in the thermodynamic limit. One easily finds the leading order contribution to be $-F_\infty = \ln R\mathcal{F}$, where in the phase with broken replica symmetry we have for $p = 1, 2, \dots, K - 1$

$$-\mathcal{F}/T = \nu_p + \frac{2}{T} \sum_{i=1}^p (\nu_{i-1} - \nu_i) T_i + \frac{1}{T^2} \sum_{i=p+1}^K (\nu_{i-1} - \nu_i) T_i^2, \quad T_{p+1} < T < T_p, \quad (46)$$

and finally for $p = K$

$$-\mathcal{F}/T = \nu_k + \frac{2}{T} \sum_{i=1}^K (\nu_{i-1} - \nu_i) T_i, \quad 0 < T < T_K = g_K. \quad (47)$$

The corresponding replica symmetric expression valid for $T > T_1 = T_c$ is given by

$$-\mathcal{F}/T = 1 + \frac{1}{T^2} \sum_{i=1}^K (v_{i-1} - v_i) T_i^2. \quad (48)$$

Interestingly, these expressions reproduce exactly, *mutatis mutandis* the leading order free-energy expressions of Derrida’s GREM [8], with a particularly clear interpretation in terms of particle localization inside smaller and smaller blobs as the temperature is reduced. Corrections to the free energy can also be found, but the corresponding expressions are rather cumbersome and are not universal but model dependent. In appendix B, we provide the explicit free-energy expression for any value of R for a different model with K -step RSB, which has the same GREM-like thermodynamic limit as the present model.

We end this section by a comment on the nature of that latter model, which we believe deserves separate mentioning. Disordered Hamiltonians usually analysed in spin-glass literature give rise to either $K = 1$ (1RSB), or to $K = \infty$ (FRSB) Parisi patterns, see, for example, the results in the framework of the so-called spherical model of spin-glasses in [33, 34]. As is easy to show, see [35] and also [10], the general class of the models of the type (1) includes the spherical model as a special case. It is therefore can be of some independent interest to provide an explicit example of a system of this sort which has a K -step version of the Parisi hierarchy as an *exact* solution, for arbitrary $K \geq 1$ (see [36] for a recent example of a model with a $K = 2$ RSB solution). In appendix B, we succeed in constructing such an example in the framework of the model (1) of a particle in a random potential when $N \rightarrow \infty$ even for *finite values* of the sample radius $R < \infty$ (this case can be looked at as corresponding to a *bona fide* p-(soft)spin model with a spherical constraint). It comes as no surprise that in the appropriately taken thermodynamic limit $R \rightarrow \infty$ this type of models reproduce again the same GREM behaviour as elsewhere in the present paper.

3.3. Multifractality of the Boltzmann–Gibbs measure

Important information about structure of the Gibbs–Boltzmann equilibrium measure $p_\beta(\mathbf{r}) = \frac{1}{Z(\beta)} \exp -\beta V(\mathbf{r})$, $\beta = \frac{1}{T}$ can be extracted from the knowledge of moments

$$m_q = \int_{|\mathbf{r}| \leq L} p_\beta^q(\mathbf{r}) \, d\mathbf{r} = \frac{Z(\beta q)}{[Z(\beta)]^q}. \quad (49)$$

In the thermodynamic limit of the sample volume $V_L \rightarrow \infty$ one expects typically

$$m_q \sim V_L^{-\tau_q}, \quad (50)$$

where the set of exponents τ_q reflects the spatial organization of the Gibbs–Boltzmann weights. For example, if the weights are of the same order of magnitude across the sample volume, the normalization condition implies locally $p_\beta(\mathbf{r}) \sim V_L^{-1}$, and a simple power counting predicts the exponents $\tau_q = q - 1$. In such a situation it is conventional to speak about a *delocalized* measure. The opposite case of a fully *localized* measure describes the situation when essential Gibbs–Boltzmann weights concentrate in the thermodynamic limit in a domain with the finite total volume $V_\xi \ll V_L \rightarrow \infty$, and are vanishingly small outside that domain. This situation is obviously characterized by trivial exponents $\tau_{q>0} = 0$ and $\tau_{q<0} = \infty$. Finally, in many interesting situations the exponents τ_q may depend on q nonlinearly, and in this case one commonly refers to the *multifractality* of the measure. Equations (49) and (50) imply the following expression for the characteristic exponents τ_q in the general case:

$$\tau_q = |q| \beta \mathcal{F}(|q|\beta) - q \beta \mathcal{F}(\beta), \quad (51)$$

relating them to the appropriately normalized free energy of the system

$$\mathcal{F}(\beta) = - \lim_{V_L \rightarrow \infty} \frac{\ln Z(\beta)}{\beta \ln V_L}. \quad (52)$$

An alternative way of characterizing multifractality invokes the so-called singularity spectrum function $f(\alpha)$. This function characterizes the number $dN(\alpha) = V_L^{f(\alpha)} d\alpha$ of sites in the sample where the local Gibbs–Boltzmann measure scales as $p_\beta(\mathbf{r}) \sim V_L^{-\alpha}$ in the thermodynamic limit. The definition allows us to extract the characteristic exponents τ_q as

$$\tau_q = - \lim_{V_L \rightarrow \infty} \frac{\ln \int_{f(\alpha) \geq 0} e^{-\ln V_L [\alpha q - f(\alpha)]} d\alpha}{\ln V_L}. \quad (53)$$

Note, that the restriction of the integration range by the condition $f(\alpha) \geq 0$ is necessary in order to remove the rare events found in the vanishing number $dN(\alpha) \rightarrow 0$ of sites in the thermodynamic limit[13]. This indeed ensures that the extracted exponents τ_q characterize the typical behaviour of the moments. Performing the α -integration by the Laplace method one finds that the two ways of characterizing multifractality, by the set of exponents τ_q or by the singularity spectrum $f(\alpha)$, turn out to be simply related by a Legendre transform: $\tau_q = \alpha_* q - f(\alpha_*)$, $q = f'(\alpha_*)$.

The expressions for the free energy calculated in the previous section allow us to extract the multifractality exponents τ_q from equation (51)–(52) and to investigate the corresponding singularity spectrum, see the relation equation (53). For the systems with a finite number K of levels of hierarchy the singularity spectrum $f(\alpha)$ turns out to be piecewise parabolic, generalizing earlier results obtained in the framework of REM-like model with a single-scale logarithmic correlations [9, 11]. For the case of an infinite hierarchy of scales in the landscape, i.e. for a continuous set of exponents ν , the behaviour turns out to be rather rich and unusual. Below we give a short account of the results for the example of continuous function $\Phi'(y)$ assuming $\Phi'(0) = 0$ for simplicity.

The associated singularity spectrum $f(\alpha)$ calculated via equation (53) is positive in an interval $\alpha \in (\alpha_{\min}, \alpha_{\max})$. The positions of the zeros $\alpha_{\min}, \alpha_{\max}$ of the function $f(\alpha)$ are given by

$$\alpha_{\min} = -\beta \mathcal{F}(\beta) - 2\beta \int_0^1 \sqrt{\Phi'(y)} dy, \quad (54)$$

$$\alpha_{\max} = -\beta \mathcal{F}(\beta) + 2\beta \int_0^1 \sqrt{\Phi'(y)} dy. \quad (55)$$

The function $f(\alpha)$ is symmetric with respect to the midpoint of the interval of interest, $\alpha_m = (\alpha_{\min} + \alpha_{\max})/2 = -\beta \mathcal{F}(\beta) > 0$, where it has the maximum $f(\alpha_m) = 1$ as expected. Close to this maximum, namely, in the subinterval $\alpha \in (\alpha_-, \alpha_+)$, where the endpoints $\alpha_\pm = \alpha_m \pm 2A(\beta) \frac{T}{T_c}$ with $A(\beta) = \beta^2(\Phi(1) - \Phi(0))$ the singularity spectrum has a simple parabolic shape

$$f(\alpha) = 1 - \frac{1}{4A(\beta)}(\alpha - \alpha_m)^2, \quad \alpha_- \leq \alpha \leq \alpha_+. \quad (56)$$

In particular, at the boundaries $f(\alpha_\pm) = 1 - \beta_c^2(\Phi(1) - \Phi(0))$. Note that in the REM-like limit $\Phi(y) = g^2 y$ we have $\alpha_{\min/\max} \rightarrow \alpha_{-/+}$ and the parabolic behaviour extends to the whole interval of positivity of $f(\alpha)$, in full agreement with the results of [11, 9].

At the same time for $\alpha \notin (\alpha_-, \alpha_+)$ the GREM-like model may show a much richer multifractal structure manifesting itself, in particular, via a quite unusual shape of the

singularity spectrum close to the zeros $\alpha_{\min}, \alpha_{\max}$, equation (54). To illustrate this fact, we consider a broad class of functions $\Phi(y)$ behaving at small arguments $y \ll 1$ as $\Phi(y) \approx C^2 y^{2s+1}$, where $s \geq 0$ and the coefficient $0 < C < \infty$. In particular, in the limiting case $s \rightarrow 0$ we are back to the old REM-like model. The behaviour of $f(\alpha)$ in the vicinity of the endpoints α_{\min} or α_{\max} is dictated by the behaviour of the multifractality exponents at large q , i.e. $\tau_{|q| \rightarrow \infty}$. As is easy to see from equation (51) this asymptotics is in turn extracted from the knowledge of the low-temperature behaviour of the free energy: $\beta \mathcal{F}(\beta)|_{\beta \rightarrow \infty}$. To investigate that limit from equation (32) we need to know the low-temperature asymptotics of the parameter ν_* related to T via equation (29). In this way we find that for the chosen $\Phi(y)$ the parameter $\nu_*(T)$ behaves as $\nu_*(T \rightarrow 0) \approx \left(\frac{T}{C\sqrt{2s+1}}\right)^{-1/s}$. After simple algebra we find with the required accuracy

$$-\beta \mathcal{F}(\beta)|_{\beta \rightarrow \infty} \approx 2\beta \int_0^1 \sqrt{\Phi'(y)} dy - \alpha_c(\beta), \tag{57}$$

where we introduced the short-hand notation

$$\alpha_c(\beta) = \frac{2s^2}{(s+1)(2s+1)} (\beta C \sqrt{2s+1})^{-\frac{1}{s}}. \tag{58}$$

Now the relation equation (51) immediately yields the required asymptotic behaviour of the multifractal exponents:

$$\tau_q = \begin{cases} q\alpha_{\min} - \alpha_c(\beta)q^{-\frac{1}{s}}, & \text{for } q \rightarrow \infty \\ q\alpha_{\max} - \alpha_c(\beta)|q|^{-\frac{1}{s}}, & \text{for } q \rightarrow -\infty, \end{cases} \tag{59}$$

where $\alpha_{\min/\max}$ are precisely the zeroes of $f(\alpha)$ given by equation (54). We thus see that the singular behaviour revealed by (59) is very different from the situation typical for other types of disordered systems [12, 13] where one always observes a *precise* linear behaviour $\tau_q = q\alpha_{\min/\max}$ starting from some value of $|q|$ (see formula (2.42) in [13] and discussions around it). In particular, such anomalous asymptotics is translated by the Legendre transform to the anomalous singularity spectrum behaviour close to the left and right zero,

$$f(\alpha) \approx \frac{s+1}{s^{s/(s+1)}} \alpha_c(\beta)^{s/(s+1)} |\alpha - \alpha_{\min/\max}|^{\frac{1}{s+1}}. \tag{60}$$

We see that for any $s > 0$ the derivative of the singularity spectrum diverges when approaching zeroes as $f'(\alpha) \sim |\alpha - \alpha_{\min/\max}|^{-\frac{s}{s+1}} \rightarrow \infty$. Note that this is again very different from the standard behaviour observed in other disordered systems where always $f'(\alpha) < \infty$ at zeros of $f(\alpha)$ [13].

4. Discussion and conclusion

In this last section, we want to discuss several aspects of our model which are certainly worth investigating further, among which: (i) the case of finite dimensions; (ii) the dynamics in such a multiscale landscape and (iii) the relation with multifractal random walks.

4.1. Multiscale logarithmic landscape in finite dimensions

As detailed at the end of section 2.2, the exact, $N \rightarrow \infty$ results for the single-scale logarithmic potential match perfectly with the results obtained by Carpentier and Le Doussal using RG, numerical and heuristic methods [9] in finite dimensions. There is no reason to doubt that these results are in fact exact in *all* dimensions $N \geq 1$. Although our model is substantially more involved, we see that essentially the same physical mechanisms are at play in both models,

in particular in the case of a finite number K of hierarchies. Therefore, it is very tempting to conjecture that the GREM behaviour revealed by above in the infinite-dimensional setting should also hold in *all* spatial dimensions, down to $N = 1$. It would be very interesting to see if the corresponding RG and travelling wave formalism can be generalized to support this conclusion. In the case of finite K , this looks indeed quite feasible.

If this conjecture is true, we would then have indeed explicitly constructed a Parisi landscape in finite dimensions fully in terms of *stationary* Gaussian processes. How do we reconcile this with the ultrametric properties of the Parisi construction? Consider the following distance D_R defined for any two points \mathbf{r}, \mathbf{r}' inside a sphere of the radius R in the Euclidean space of any dimension N ,

$$D_R(\mathbf{r}, \mathbf{r}') = \frac{\ln [|\mathbf{r} - \mathbf{r}'|^2 + a^2]}{2 \ln R}, \quad 0 < |\mathbf{r}|, \quad |\mathbf{r}'| \leq R. \quad (61)$$

Parameterizing $|\mathbf{r}| \equiv R^{\alpha(\mathbf{r})}$, $0 \leq \alpha \leq 1$, we see that in fact

$$\lim_{R \rightarrow \infty} D_R(\mathbf{r}, \mathbf{r}') = \max\{\alpha(\mathbf{r}), \alpha(\mathbf{r}')\}. \quad (62)$$

One can easily check that the latter function used as a distance converts the Euclidean sphere into a so-called ultrametric space: every triangle will have at least two sides equal. We thus conclude that our choice of the model corresponds to insisting that the covariance of the random potential values in two points in space should depend only on the ultrametric distance inside our growing sphere. The original construct of the GREM by Derrida in fact proceeds in a similar way, and therefore the coincidence between our model of the GREM could have been, with hindsight, anticipated. Indeed, Derrida started with 2^M random variables, attached to the vertices of a hypercube of 2^M points supplied with a tree structure and associated ultrametric distance [31]. The Gaussian random energies used for constructing the Boltzmann weights were built from those ingredients in such a way that their correlation function depended only on that ultrametric distance. Although the methods used in this paper have very little in common with techniques used by Derrida, or other authors analysing GREM, the similarities between the two problems are apparent. This line of reasoning suggests that the convergence to the GREM limit could be indeed expected by invoking universality arguments. A remarkable recent progress [31] based on the idea of Ruelle probability cascades [37] has allowed one to analyse Derrida's original GREM in full mathematical rigour. It would be extremely interesting to see whether the Euclidean version of the GREM analysed in the present paper is amenable to a similar kind of rigorous analysis, without any reference to the replica trick, powerful heuristically but still ill-defined mathematically.

Another, more speculative aspect of the problem is also worth mentioning here. In a recent work, Moore [38], argued that low-temperature phases with 1RSB fail to survive in finite spatial dimensions. This, in Moore's argument, is intimately related to a generic absence of marginally stable modes in a 1RSB fluctuation spectrum. To this end, the stability of the 1RSB low-temperature phase in the present model (1) was investigated in [10], and found to be controlled by two eigenmodes, denoted in [10] as Λ_0^* and Λ_K^* (see equations (B.29) and (B.30) of appendix B of [10]). Generically those two were demonstrated to be positive, but for the logarithmic potential (8) both Λ_0^* and Λ_K^* identically vanish everywhere in the low-temperature phase. That property makes the associated 1RSB phase *marginally stable*⁹. Thus, the glass behaviour in the case of logarithmic correlations could survive for finite N due to marginality of the fluctuation spectrum. This picture would be indeed in agreement with the above-discussed RG results of [9]. We believe that similar marginality of fluctuations

⁹ This fact, though not explicitly mentioned in [10], immediately follows from definitions (B.29) and (B.30) after substituting for $q_1 - q_0 = Q$ and $q_d - q_1 = y$ the expressions (74) and (79) of that paper.

spectra should exist also in the general K -step case of our model, and is related to the fact of vanishing Schwarzian derivative, exactly or asymptotically in the thermodynamic limit. The corresponding calculation looks cumbersome, but is certainly feasible, see similar work in [39], and is yet to be done. In general, any results in these directions are highly desirable, with an ultimate goal of performing perturbative expansions around $N = \infty$ limit which remains to be one of outstanding challenging tasks.

4.2. Diffusion in a multiscale logarithmic landscape

The rich behaviour found in the thermodynamics of a single particle in a random potential also has interesting dynamical counterparts. In the infinite dimension limit, the problem of a Langevin particle in a random potential has been solved in details by Franz and Mézard [40] and Cugliandolo and Le Doussal [35], for both the short-range and the long-range cases. These results reveal long-time relaxation, aging and other effects typical of glassy dynamics. The marginal (monoscale) logarithmic case was however not specifically studied in these papers. For finite N , this marginal case was studied in [42, 43] using RG methods. The main quantity of interest is the dynamical exponent z , defined as

$$\Delta^2(t) = \langle (\mathbf{r}(t+t_0) - \mathbf{r}(t_0))^2 \rangle \sim_{t \rightarrow \infty} t^{2/z}. \quad (63)$$

The result, conjectured to hold at all orders in perturbation theory and for any N , is that z varies continuously with the strength of the disorder g and temperature T ,

$$z = 2 + 2 \left(\frac{g}{T} \right)^2. \quad (64)$$

When $g = 0$, one recovers the diffusion exponent $z = 2$, as it should. It was noted by Castillo and Le Doussal [41] that this result cannot hold down to $T = 0$ in $N = 1$ dimension, because it violates exact bounds. It was argued instead that below the static transition $T_c = g$ found in section 2.2, the exponent in fact is modified and reads $z = 4T_c/T$. Therefore the static transition is also a dynamical transition in $N = 1$.

The situation in higher dimensions is unclear, in particular it is not immediately clear that the dynamical transition where equation (64) ceases to hold still coincides with the static transition. This is another motivation for studying in details the limit $N \rightarrow \infty$. However, it seems reasonable to conjecture that for the multiscale landscape model at high enough temperature, the diffusion exponent becomes scale dependent. For a finite number K of hierarchies, and for $T > T_1 \equiv T_c$, we expect to find

$$z(\Delta) = 2 + 2 \left(\frac{T_p}{T} \right)^2, \quad (65)$$

where T_p is the critical temperature associated with ν_p such that $R^{\nu_p} < \Delta < R^{\nu_{p-1}}$. At temperatures lower than T_c , it would appear natural to conjecture that all levels such that $T < T_p$ follow the Castillo–Le Doussal scenario, whereas levels such that $T_p > T$ are still ruled by the above exponent. However, the situation might be more complicated because the possibility of aging at low temperatures seems to have been overlooked in [41]. In high enough dimensions, the analogy with the GREM suggests that the multiscale logarithmic model of the present paper might be a real-space realization of the multi-level trap model introduced and studied in [44]. We would then witness a very rich dynamical behaviour, where all levels such that $T < T_p$ age (concerning large length scales), whereas small length scales, such that $T > T_p$, are still stationary [44, 45]. It would be extremely interesting to study these aspects in more details.

4.3. A generalized multifractal random walk

The monoscale logarithmic landscape model in $N = 1$ has in fact deep connections with the multifractal random walk (MRW) construction of Bacry, Muzy and Delour [14, 46]. If one treats the coordinate r as a time variable t , and call X_t the position of a random walk at time t , the BMD model is defined by the following evolution equation:

$$dX_t = mdt + \sigma_t dW_t \quad \sigma_t = \sigma_0 \exp \beta V_t, \tag{66}$$

where dW is the usual Brownian process, m a drift and V_t is the logarithmically-correlated Gaussian process considered above (possibly shifted such that $\langle e^{2\beta V_t} \rangle_V = 1$). It was shown by BMD that the resulting process is multifractal in the time regime $a \ll t \ll R$. For $m = 0$, all odd moments vanish and even moments are given by [14],

$$\langle (X_t - X_{t+\tau})^n \rangle = M_n \tau^{\zeta_n} \quad \text{with} \quad \zeta_n = \frac{n}{2} - (\beta g)^2 \frac{n(n-2)}{2}, \tag{67}$$

as long as $n < (\beta g)^{-2}$ beyond which all moments diverge: the distribution of increments $\Delta = (X_t - X_{t+\tau})$ has a power-law tail $\Delta^{-1-\mu}$ with an exponent given by

$$\mu = \left(\frac{T}{T_c} \right)^2. \tag{68}$$

The connection with the thermodynamical problem addressed above is clear: the MRW is a standard random walk subordinated to a stochastic time s_t , such that,

$$s_{t+\tau} - s_t = \int_t^{t+\tau} du e^{2\beta V_u}, \tag{69}$$

which is a restricted partition function at temperature $T/2$.

The generalization to a multiscale logarithmic process is very natural. Taking $V_t = \sum_{v=1}^K V_{v,t}$ as in equation (14) above, one constructs a generalized MRW with, in the limit $R \rightarrow \infty$, K different epochs such that $v_p < \ln \tau / \ln R < v_{p-1}$ such that the moments of the walk are given by the above multifractal scaling, equation (67), but with a different multifractal spectrum,

$$\zeta_n^{(p)} = \frac{n}{2} - \left(\frac{T_p}{T} \right)^2 \frac{n(n-2)}{2}. \tag{70}$$

This formula can be given a meaning in the continuous limit as well. In this case the exponents $\zeta_n^{(p)}$ acquire a continuous dependence on $\ln \tau$: they are still given by equation (70) but with the ratio T/T_p replaced with the Parisi function $X(\nu)$, $\nu = \ln \tau / \ln R$. It would be interesting to understand in more details the extreme value statistics of the GMRW, following the method introduced in [9, 47].

4.4. Final remarks

The model we have introduced appears to contain a very rich phenomenology, and deserves in our opinion further investigations. Many points would require a more rigorous mathematical investigation, concerning in particular the finite-dimensional version of the model and its dynamical properties, in particular in the continuous hierarchy case $K \rightarrow \infty$. We believe that our construction sheds an important light on the understanding of replica symmetry breaking: clearly the fact that Parisi's ultrametric scheme is relevant for a translationally invariant, Gaussian potential in finite (even one) Euclidean dimension is very satisfying, if only for pedagogical reasons. A better intuition on the Parisi solution for the SK model for a large but finite number of spins M , and the recently reported scaling $K \sim M^{1/6}$ [48], might also be within reach.

Acknowledgments

This paper is dedicated to David Sherrington on the occasion of his 65th birthday. The model studied here is of course in close filiation with the ideas and methods David contributed to develop and popularize. An essential part of this research was performed during the first author’s stay at the Institute of Theoretical Physics, Cologne University, Germany. YF appreciates kind hospitality extended to him during that period, as well as the financial support from the Alexander von Humboldt foundation through Bessel Research Award. The research in Nottingham was supported by grant EP/C515056/1 from EPSRC (UK). This project was initiated during the workshop on random matrix theory held in Jagellonian University, Cracow, May 2007. We thank the organizers for this opportunity. We also want to thank Mike Moore for interesting discussions.

Appendix A. Calculation of the overlap function, equation (12)

In the framework of the replica trick we represent the normalization factor in the product of the Boltzmann–Gibbs weights as $1/Z^2 = \lim_{n \rightarrow 0} Z^{n-2}$. This trick allows the disorder average to be performed explicitly, and (12) takes the form

$$\begin{aligned} \pi(\mathcal{D}) &= \lim_{n \rightarrow 0} \int_{|\mathbf{r}_a| \leq L, \forall a} \langle e^{-\beta \sum_{a=1}^n V(\mathbf{r}_a)} \rangle_V \delta \left(\mathcal{D} - \frac{1}{2N} [\mathbf{r}_1 - \mathbf{r}_2]^2 \right) \prod_{a=1}^n d\mathbf{r}_a \\ &= \lim_{n \rightarrow 0} e^{\frac{\beta^2}{2} N n f(0)} \int_{|\mathbf{r}_a| \leq R\sqrt{N}} e^{N\beta^2 \sum_{a < b} f(\frac{1}{2N}(\mathbf{x}_a - \mathbf{x}_b)^2)} \delta \left(\mathcal{D} - \frac{1}{2N} [\mathbf{r}_1 - \mathbf{r}_2]^2 \right) \prod_{a=1}^n d\mathbf{r}_a. \end{aligned}$$

At the next step we exploit the $O(N)$ rotational invariance of the integrand and, assuming $N > n$, introduce the scalar products $\mathbf{r}_a \mathbf{r}_b = q_{ab}$ as new integration variables, see [10] for a general description of the method. After further rescaling $q_{ab} \rightarrow Nq_{ab}$ the integral in the right-hand side of the equation takes the form

$$\mathcal{C}_{N,n} N^{Nn/2} \int_{D_Q} (\det Q)^{-(n+1)/2} e^{-N\beta\Phi_n(Q)} \delta \left(\mathcal{D} - \frac{1}{2}(q_{11} + q_{22}) + q_{12} \right) dQ, \tag{A.1}$$

where

$$\Phi_n(Q) = -\frac{1}{2\beta} \ln(\det Q) - \beta \sum_{a < b} f \left[\frac{1}{2}(q_{aa} + q_{bb}) - q_{ab} \right] \tag{A.2}$$

and the integration domain D_Q over the matrix Q with entries q_{ab} is already N -independent: $D_Q = \{Q \geq 0, q_{aa} \leq R^2, a = 1, \dots, n\}$. The proportionality constant $\mathcal{C}_{N,n}$ is also known.

The form of the integrand in equation (A.1) is precisely the one required for the possibility of evaluating the replicated partition function in the limit $N \rightarrow \infty$ by the Laplace method. Taking into account the permutational symmetry of the integrand with respect to the replica indices, we find after a straightforward calculation

$$\pi(\mathcal{D}) = \lim_{n \rightarrow 0} \frac{1}{n(n-1)} \sum_{a \neq b} \delta \left(\mathcal{D} - \frac{1}{2}(q_{aa} + q_{bb}) + q_{ab} \right) \Big|_{Q \in \text{stationary point}}. \tag{A.3}$$

To perform the replica limit explicitly we rely upon the assumption of validity of the hierarchical ansatz for the stationary-point solution suggested by Parisi, see a description in the context of the present model in appendix A of [10]. Denoting m_l the size of the blocks

in the Parisi scheme, we find in the standard way (cf equation (46) of [10]),

$$\pi(\mathcal{D}) = \lim_{n \rightarrow 0} \sum_{l=0}^k (m_{l+1} - m_l) \delta(\mathcal{D} - q_d + q_l) = \int_0^{q_d} \delta(\mathcal{D} - q_d + q) x'(q) dq = x'(q_d - \mathcal{D}), \tag{A.4}$$

where $x(q)$ is a non-trivial non-decreasing Parisi function characterizing the low-temperature phase. In the present model, it turns out that the parameter q_d is related to the effective radius of the sample as $q_d = R^2$. It is actually more convenient to use the variable $d = R^2 - q$ rather than q itself, as we do in the main body of the paper. Accordingly, we pass from $x(q)$ to the non-increasing function $x(d)$ to which $\pi(\mathcal{D})$ is especially simply related,

$$\pi(\mathcal{D}) = -\frac{d}{d\mathcal{D}} x(\mathcal{D}), \tag{A.5}$$

which is the counter-part of the standard interpretation of $x(q)$ in terms of the overlap probability in spin-glasses ([49]).

The above construction is valid for any choice of random Gaussian potential with any covariance function of the form equation (1). For the case of multiscale logarithmic potential studied in the present paper the argument d of the function $x(d)$ generically covers the interval $d \in [d_{\min}, d_{\max}]$, with $d_{\min} \sim R^{2\nu_*}$ and $d_{\max} \sim R^2$, where the value ν_* determined for a given temperature T by equation (29). Accordingly, it is natural to pass from the Parisi function $x(d)$ to its counterpart $X(\nu)$, $\nu \in [\nu_*, 1]$ by replacing $d \rightarrow R^{2\nu}$. As is clear, ν is precisely the ‘ultrametric’ separation between the two positions related to the scaled squared Euclidean distance \mathcal{D} as $\nu = \frac{1}{2 \ln R} \ln \mathcal{D}|_{R \rightarrow \infty}$ (cf equation (62)). It is then immediate to check that the probability $\Pi(\nu)$ for two independent particles to end up at an ultrametric separation ν is related to the function $X(\nu)$ in precisely the same way as $\pi(\mathcal{D})$ to $x(d)$,

$$\Pi(\nu) = -2 \ln R \frac{d}{d \ln(\mathcal{D})} x(\mathcal{D})|_{\mathcal{D}=R^{2\nu}} = -\frac{d}{d\nu} X(\nu). \tag{A.6}$$

Appendix B. A model for K -step replica symmetry breaking

The idea of the construction comes from an observation that the most general function $f(u)$ with locally vanishing Schwarzian derivative $S(u) = 0$, see equation (7), is given by $f(u) = f - Au - g^2 \ln(u + a^2)$. Apparently, linear in u term violates the requirement $f'(u \rightarrow \infty) \rightarrow 0$ and for this reason was discarded by us earlier. Now we can try, by choosing constants f and A judiciously, to construct a covariance function $f(u)$ globally from the above local patches in such a way, that the composite solution will have the vanishing $S(u)$ everywhere, except in a finite number of points. As the derivative $x'(d)$ of the Parisi order-parameter function from equation (22) is proportional to the Schwarzian derivative $S(d)$, this construction should provide us with a piecewise-constant order-parameter function required for the Parisi pattern with K levels of hierarchy. For the sake of simplicity we discuss below in detail the simplest non-trivial case $K = 2$. The generalization to arbitrary $K \geq 2$ will be apparent.

We find it more convenient to work with the structure function $\phi(u) = f(0) - f(u)$ rather than with $f(u)$ itself. The structure function obviously vanishes at the origin, providing an additional condition $\phi(0) = 0$ used to specify all the constants in our construction uniquely.

Consider the model of a particle in a Gaussian random potential with the structure function taken in the form

$$-\phi(u) = \begin{cases} f_1 - A_1 u - t_1^2 \ln(u + a_1^2), & u \geq u_* \\ f_2 - A_2 u - t_2^2 \ln(u + a_2^2), & 0 \leq u \leq u_*, \end{cases} \quad (\text{B.1})$$

where the positive parameters t_1, t_2, a_1, a_2 are considered to be given, and chosen satisfying the inequalities,

$$t_1 > t_2 \quad \text{and} \quad \frac{a_1^2}{t_1} > \frac{a_2^2}{t_2}. \quad (\text{B.2})$$

In contrast, the value of the ‘breakpoint’ u_* , as well of the constants f_1, f_2, A_1, A_2 are unknown and should be specified in terms of $a_{1,2}$ and $t_{1,2}$. Note that for $\phi(u)$ to have the meaning of a structure function requires $A_{1,2}$ to be non-negative, see equation (3).

As we are to use such a function for building the Parisi order-parameter $x(d)$ according to equation (22), and such $x(d)$ can take only finite values between 0 and 1, one has to ensure that $\phi(u)$ is continuous together with at least two derivatives everywhere, including the breakpoint u_* . The requirement of continuity of $\phi'(u)$ at u_* immediately fixes the value of the breakpoint,

$$u_* = \frac{\frac{a_1^2}{t_1} - \frac{a_2^2}{t_2}}{\frac{1}{t_2} - \frac{1}{t_1}}. \quad (\text{B.3})$$

The consistency of the procedure requires $u_* > 0$ which is precisely the case due to the imposed conditions equation (B.2).

In the same way the continuity of $\phi'(u)$ at u_* given by (B.3) allows us to relate A_1 to A_2 via

$$A_2 = A_1 + \frac{(t_1 - t_2)^2}{a_1^2 - a_2^2}. \quad (\text{B.4})$$

At the next step the continuity of $\phi(u)$ at u_* relates f_1 to f_2 as

$$f_1 = f_2 + \frac{a_2^2 t_1 - a_1^2 t_2}{a_1^2 - a_2^2} (t_1 - t_2) + t_1^2 \ln t_1 - t_2^2 \ln t_2 + (t_1^2 - t_2^2) \ln \frac{a_1^2 - a_2^2}{(t_1 - t_2)}. \quad (\text{B.5})$$

The constant f_2 can be found from the condition $\phi(0) = 0$, which fixes $f_2 = t_2^2 \ln a_2^2$. Finally one should have in mind that although for a finite sample $x < R^2$, to have a well-defined model for any $x < \infty$ we should impose the condition $\phi'(u \rightarrow \infty) = 0$ as elsewhere in the paper. This necessarily implies $A_1 = 0$, and fixes all the constants of the model uniquely. Note that A_2 is indeed positive in view of equation (B.4) and the inequality $a_1^2 > a_2^2$.

Having specified the structure function $\phi(u)$, hence the covariance $f(u)$, we now can repeat the procedure of finding d_{\min}, d_{\max} , and hence the free energy of the model. Assuming $d_{\max} > u_*$ we can solve the second of equations equation (21) and find $d_{\max} = \frac{R^2 - a_1^2}{2}$. Such solution is indeed the valid one for sufficiently large sample sizes satisfying $R^2 > R_m^2 = [a_1^2(t_1 + t_2) - 2a_2^2 t_1] / (t_1 - t_2)$. Assuming the condition is satisfied, the transition to the phase with broken replica symmetry occurs at the temperature found from equation (23) which is given by

$$T_c = t_1 \frac{R^2 - a_1^2}{R^2 + a_1^2}. \quad (\text{B.6})$$

Then the solution to the first of equations equation (21) reads

$$d_{\min} = \begin{cases} a_2^2 \frac{T}{t_2} / \left(1 - \frac{T}{t_2}\right), & d_{\min} < u_*, \\ a_1^2 \frac{T}{t_1} / \left(1 - \frac{T}{t_1}\right), & u_* < d_{\min}, \end{cases} \quad (\text{B.7})$$

and further requiring consistency of this solution with the expression equation (B.3) finally yields

$$d_{\min} = \begin{cases} a_2^2 \frac{T}{t_2} / \left(1 - \frac{T}{t_2}\right), & T \leq T_{\min}, \\ a_1^2 \frac{T}{t_1} / \left(1 - \frac{T}{t_1}\right), & T_{\min} \leq T < T_c, \end{cases} \quad (\text{B.8})$$

where the ‘breakpoint temperature’ T_{\min} is given by

$$T_{\min} = t_1 t_2 \frac{a_1^2/t_1 - a_2^2/t_2}{a_1^2 - a_2^2}. \quad (\text{B.9})$$

It is easy to see that indeed $t_2 < T_{\min} < T_c < t_1$ for $R > R_m$, so such a solution is consistent. Note also that $d_{\min}(T) \leq u_*$ for $T < T_{\min}$, and $d_{\min}(T_c) = d_{\max}$ as expected.

A simple calculation shows that the Parisi order-parameter function $x(d)$ in the temperature range $0 \leq T < T_{\min}$ indeed consists of two perfect steps:

$$x(d) = \begin{cases} \frac{T}{t_2} & \text{for } d_{\min} \leq d < u_*, \\ \frac{T}{t_1} & \text{for } u_* < d < d_{\max} \end{cases}, \quad (\text{B.10})$$

whereas in the range $T_{\min} < T < T_c$ the value d_{\min} exceeds the breakpoint u_* , and hence only a single step survives

$$x(q) = \frac{T}{t_1} \quad \text{for } d_{\min} < d < d_{\max}. \quad (\text{B.11})$$

Now we can easily calculate the free energy for $T < T_c$ from equation (20). In particular, as we have for $0 \leq T < T_{\min}$

$$\sqrt{f''(u)} = \begin{cases} \frac{t_2}{u + a_2^2}, & d_{\min} \leq u < u_*, \\ \frac{t_1}{u + a_1^2}, & u_* < u < d_{\max}, \end{cases} \quad (\text{B.12})$$

we find that in this range of temperatures

$$I = - \int_{d_{\min}}^{d_{\max}} [f''(u)]^{1/2} du = -t_2 \ln \left(\frac{u_* + a_2^2}{d_{\min} + a_2^2} \right) - t_1 \ln \left(\frac{d_{\max} + a_1^2}{u_* + a_1^2} \right).$$

At the same time, for $T_{\min} < T \leq T_c$ we have $\sqrt{f''(u)} = \frac{t_1}{u+a_1^2}$ so that

$$I = -t_1 \ln \left(\frac{d_{\max} + a_1^2}{d_{\min} + a_1^2} \right).$$

The free-energy equation (20) is then given by

$$-\frac{F_{\infty}}{T} = \ln \sqrt{2\pi e} - \frac{f_1}{2T^2} + \frac{1}{2} \ln \frac{T}{t_1} + \frac{t_1}{T} \left[\frac{1}{2} + \ln \frac{R^2 + a_1^2}{2} \right] + \frac{1}{2} \left(1 + \frac{t_1^2}{T^2} - \frac{2t_1}{T} \right) \ln \frac{a_1^2}{1 - \frac{T}{t_1}}, \quad T_{\min} \leq T \leq T_c \quad (\text{B.13})$$

and

$$-\frac{F_{\infty}}{T} = \ln \sqrt{2\pi e} - \frac{f_2}{2T^2} + \frac{1}{2} \ln \frac{T}{t_2} - \frac{t_2}{T} \left[\frac{1}{2} - \ln(u_* + a_2^2) \right] + \frac{t_1}{T} \left[\frac{1}{2} - \ln(u_* + a_1^2) \right] + \frac{t_1}{T} \left[\frac{1}{2} + \ln \frac{R^2 + a_1^2}{2} \right] + \frac{1}{2} \left(1 + \frac{t_2^2}{T^2} - \frac{2t_2}{T} \right) \ln \frac{a_2^2}{1 - \frac{T}{t_2}}, \quad 0 \leq T \leq T_{\min}. \quad (\text{B.14})$$

One can check that at the breakpoint temperature $T = T_{\min}$ the free energy is continuous.

Finally, for $T > T_c$ the replica-symmetric expression for d_s obtained from equation (24) is given by

$$d_s = \frac{1}{2(1+t_1^2/T^2)} [R^2 - a_1^2 + \sqrt{(R^2 + a_1^2)^2 + 4R^2 a_1^2 t_1^2 / T^2}] \quad (\text{B.15})$$

and the free energy is obtained by substituting this into equation (24), and remembering $d_s > u_*$. One can check that $d_s|_{T \rightarrow T_c} = (R^2 - a_1^2)/2 = d_{\max}$, and the free energy is again continuous at the transition temperature T_c .

All the above expressions were derived for finite sample sizes, provided $R > R_m$. Let us now investigate for this model the thermodynamic limit $R \rightarrow \infty$, taken at a fixed temperature T . As elsewhere in the paper, we assume the scaling $a_1 = R^{\nu_1}$, $a_2 = R^{\nu_2}$, with $0 < \nu_2 < \nu_1 < 1$, which implies $a_1 \gg a_2$ in the thermodynamic limit. We immediately find that the critical temperature tends to $T_c = t_1$, and $d_{\max} \rightarrow R^2/2$, $d_s \rightarrow R^2/(1 + T_c^2/T^2)$. Similarly, the breakpoint temperature $T_{\min} \rightarrow t_2$, and the breakpoint itself is given by the limiting value $u_* = R^{2\nu_1}/(T_c/T_{\min} - 1)$. We also need to know that $f_2 = 2\nu_2 T_{\min}^2 \ln R$ and $f_1 = 2[\nu_1(T_c^2 - T_{\min}^2) + \nu_2 T_{\min}^2] \ln R$. The leading contributions to the free energy found from equations (B.13) and (B.14) are given by

$$-\frac{F_\infty}{T \ln R} = \begin{cases} \nu_2 + \frac{2T_{\min}}{T}(\nu_1 - \nu_2) + \frac{2T_c}{T}(1 - \nu_1), & 0 \leq T \leq T_{\min} \\ \nu_1 + \frac{2T_c}{T}(1 - \nu_1) + (\nu_1 - \nu_2) \frac{T_{\min}^2}{T^2}, & T_{\min} \leq T \leq T_c \\ 1 + (1 - \nu_1) \frac{T_c^2}{T^2} + (\nu_1 - \nu_2) \frac{T_{\min}^2}{T^2}, & T \geq T_c. \end{cases} \quad (\text{B.16})$$

These expressions are identical to $K = 2$ case of GREM free energy (46) and (48), upon identification $T_{\min} \equiv T_2$, $T_c \equiv T_1$, showing that the two models are indeed in the same universality class in the thermodynamic limit. The corrections are however already model dependent.

We further note that the parameter d_{\min} given by equation (B.7) develops in the limit $R \rightarrow \infty$ characteristic divergencies at the breakdown temperature T_{\min} as well as at T_c (cf (43)),

$$d_{\min} = \begin{cases} \frac{T}{T_{\min}} R^{2\nu_2} / \left(1 - \frac{T}{T_{\min}}\right), & 0 \leq T < T_{\min}, \\ \frac{T}{T_c} R^{2\nu_1} / \left(1 - \frac{T}{T_c}\right), & T_{\min} < T < T_c. \end{cases} \quad (\text{B.17})$$

The corrections to the free energy however do not have any divergencies as the logarithmic terms come with vanishing prefactors.

The $K = 2$ construction discussed above has obvious generalization to any $K \geq 2$. We choose the structure function in the form

$$-\phi(u) = f_p - A_p u - t_p^2 \ln(u + a_p^2), \quad u_*^{(p)} \leq u \leq u_*^{(p-1)}, \quad p = 1, 2, \dots, K, \quad (\text{B.18})$$

where the positive parameters t_1, \dots, t_K ; a_1, \dots, a_K are given, and chosen satisfying the inequalities,

$$t_1 > t_2 > \dots > t_K \quad \text{and} \quad \frac{a_1^2}{t_1} > \frac{a_2^2}{t_2} > \dots > \frac{a_K^2}{t_K}. \quad (\text{B.19})$$

Those inequalities will ensure that a strictly decreasing sequence of the breakpoints $u_*^{(0)} = R^2 > u_*^{(1)} > u_*^{(2)} > \dots > u_*^{(K-1)} > 0$ can be found from the conditions of continuity of

$\phi''(u)$. Requiring also continuity of $\phi'(u)$ and $\phi(u)$ relates A_p to A_{p-1} and f_p to f_{p-1} for any p . Finally we set $\phi(0) = 0$ which gives $f_K = t_K^2 \ln a_K^2$, and put $A_1 = 0$ to satisfy the behaviour at $u \rightarrow \infty$. This choice fixes all the parameters of the model uniquely, and ensures $A_p > 0$. Considering the effective radius R exceeding some minimal value $R_m \sim a_1^2$, the model will have the sequence of ‘break temperatures’ $T_c = T_1 > T_2 > \dots > T_K > 0$, with the order-parameter function consisting of l decreasing steps for $T_{l-1} < T < T_l$, with the values at those steps given by $x(d) = T/t_i$, for $u_*^{(i)} < d < u_*^{(i-1)}$, $i = 1, 2, \dots, l$, where we make a convention $u_*^{(0)} \equiv d_{\max}$, $u_*^{(l)} \equiv d_{\min}$. Taking the thermodynamic limit $R \rightarrow \infty$ with the power-law scaling of the cutoffs as $a_i = R^{2\nu_i}$ reproduces exactly the GREM equations (46) and (48).

References

- [1] Goldstein M 1969 *J. Chem. Phys.* **51** 3728
- [2] Wales D J 2004 *Energy Landscapes, with Applications to Clusters Biomolecules and Glasses* (Cambridge: Cambridge University Press)
- [3] Kurchan J and Laloux L 1996 *J. Phys. A: Math. Gen.* **29** 1929
- [4] Cavagna A 2001 *Europhys. Lett.* **53** 490
- [5] Fyodorov Y V and Williams I 2007 *J. Stat. Phys.* **129** 1081
- [6] For a recent account, see, e.g., Parisi G 2007 Mean field theory of spin glasses: statics and dynamics *Complex Systems (Les Houches Summer School)* (Amsterdam: Elsevier) (Preprint [cond-mat/0706.0094](#))
- [7] Balents L, Bouchaud J P and Mézard M 1996 *J. Phys. I (France)* **6** 1007
- [8] Derrida B 1985 *J. Phys. Lett.* **46** 401
Derrida B and Gardner E 1986 *J. Phys. C: Solid State Phys.* **19** 2253
Derrida B and Gardner E 1986 *J. Phys. C: Solid State Phys.* **19** 5783
- [9] Carpentier D and Le Doussal P 2001 *Phys. Rev. E* **63** 026110
- [10] Fyodorov Y V and Sommers H-J 2007 *Nucl. Phys. B* **764** 128
- [11] Chamon C, Mudry C and Wen X-G 1996 *Phys. Rev. Lett.* **77** 4194
Castillo H E, Chamon C C, Fradkin E, Goldbart P M and Mudry C 1997 *Phys. Rev. B* **56** 10668
- [12] Monthus C and Garel T 2007 *Phys. Rev. E* **75** 051122
- [13] Evers F and Mirlin A D 2007 *Anderson Transitions* (Preprint [0707.4378](#))
- [14] Muzy J-F, Delour J and Bacry E 2000 *Eur. Phys. J. B* **17** 537–48
Bacry E, Delour J and Muzy J-F 2001 *Phys. Rev. E* **64** 026103
- [15] Paladin G and Vulpiani A 1987 *Phys. Rep.* **156** 147
- [16] Fyodorov Y V and Bouchaud J-P 2007 *JETP Lett.* **86** 487
- [17] Yaglom A M 1987 *Correlation Theory of Stationary and Related Random Functions I* (Springer Series in Statistics) (Berlin: Springer)
- [18] Villain J, Semeria B, Lançon F and Billard L 1983 *J. Phys. C: Solid State Phys.* **16** 6153
Schulz U, Villain J, Brézin E and Orland H 1988 *J. Stat. Phys.* **51** 1
- [19] Parisi G 1990 *J. Phys. (France)* **51** 1595
Bouchaud J-P and Orland H 1990 *J. Stat. Phys.* **61** 877
- [20] Monthus C and Comtet A 1994 *J. Phys. I (France)* **4** 635
- [21] Broderix K and Kree R 1995 *Europhys. Lett.* **32** 343
- [22] Monthus C and Le Doussal P 2002 *Phys. Rev. E* **65** 066129
- [23] Mezard M and Parisi G 1990 *J. Phys. A: Math. Gen.* **23** L1229
Mezard M and Parisi G 1991 *J. Phys. I (France)* **1** 809
Mezard M and Parisi G 1992 *J. Phys. I (France)* **2** 2231
- [24] Engel A 1993 *Nucl. Phys. B* **410** 617
- [25] Monthus C and Bouchaud J-P 1996 *J. Phys. A: Math. Gen.* **29** 3847
- [26] Ben Arous G and Cerny J 2006 Dynamics of trap models Preprint [math.PR/0603344](#)
- [27] Derrida B 1981 *Phys. Rev. B* **24** 2613
- [28] Bouchaud J-P and Mézard M 1997 *J. Phys. A: Math. Gen.* **30** 7997
- [29] Comte A and Bouchaud J-P 1998 *J. Phys. A: Math. Gen.* **31** 6113
- [30] Derrida B and Spohn H 1988 *J. Stat. Phys.* **51** 817
- [31] Bovier A and Kurkova I 2004 *Ann. Inst. Henri Poincaré* **40** 439 and 481
- [32] Plotkin S S, Wang J and Wolynes P G 1996 *Phys. Rev. E* **53** 6271

- [33] Crisanti A and Sommers H-J 1992 *Z. Phys. B* **87** 341
- [34] Talagrand M 2006 *Probab. Theory Rel. Fields* **134** 339
- [35] Cugliandolo L F and Le Doussal P 1996 *Phys. Rev. E* **53** 1525
- [36] Crisanti A and Leuzzi L 2007 *Phys. Rev. B* **76** 184417
- [37] Ruelle D 1987 *Commun. Math. Phys.* **108** 225
- [38] Moore M 2006 *Phys. Rev. Lett.* **96** 137202
- [39] Carlucci D M, de Dominicis C and Temesvari T 1996 *J. Phys. I (France)* **6** 1031
- [40] Franz S and Mezard M 1994 *Physica A* **210** 48
- [41] Castillo H E and Le Doussal P 2001 *Phys. Rev. Lett.* **86** 4859
- [42] Bouchaud J-P, Comtet A, Georges A and Le Doussal P 1987 *J. Phys. (Paris)* **48** 1445
Bouchaud J-P, Comtet A, Georges A and Le Doussal P 1988 *J. Phys. (Paris)* **49** 369
Bouchaud J-P and Georges A 1990 *Phys. Rep.* **195** 127
- [43] Honkonen J and Karjalainen E 1988 *J. Phys. A: Math. Gen.* **21** 4217
- [44] Bouchaud J-P and Dean D S 1995 *J. Phys. I (France)* **5** 265
- [45] Bouchaud J-P, Dupuis V, Hammann J and Vincent E 2001 *Phys. Rev. B* **65** 024439
- [46] Borland L, Bouchaud J-P, Muzy J-F and Zumbach G 2005 The dynamics of financial markets: Mandelbrot's multifractal cascades, and beyond *Wilmott Magazine (March)* p 263 (Preprint [cond-mat/0501292](#))
- [47] Muzy J-F, Bacry E and Kozhemyak A 2006 *Phys. Rev. E* **73** 066114
- [48] Aspelmeier T, Belloire A, Marinari E and Moore M 2007 Finite size corrections in the SK model *Preprint* [cond-mat.dis-nn/0711.3445v1](#)
- [49] Mézard M, Parisi G and Virasoro M A 1986 *Spin Glasses and Beyond* (Singapore: World Scientific)